Block-Krylov Component Synthesis Method for Structural Model Reduction

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A new analytical method is presented for generating component shape vectors, or Ritz vectors, for use in component synthesis. Based on the concept of a block-Krylov subspace, easily derived recurrence relations generate blocks of Ritz vectors for each component. The subspace spanned by the Ritz vectors is called a block-Krylov subspace. The synthesis uses the new Ritz vectors rather than component normal modes to reduce the order of large, finite-element component models. An advantage of the Ritz vectors is that they involve significantly less computation than component normal modes. Both "free-interface" and "fixed-interface" component models are derived. They yield block-Krylov formulations paralleling the concepts of free-interface and fixed-interface component modal synthesis. Additionally, block-Krylov reduced-order component models are shown to have special disturbability/observability properties. Consequently, the method is attractive in active structural control applications, such as large space structures. The new fixed-interface methodology is demonstrated by a numerical example. The accuracy is found to be comparable to that of fixed-interface component modal synthesis.

Introduction

THE basic idea of component synthesis is to model individual components of a structure separately and then to couple the separate models to form an assembled system model. In the well-known component modal synthesis (CMS) methods (e.g., Refs. 1–11), each component is modeled by a set of Ritz vectors containing some type of component normal modes (eigenvectors) as a subset. Other Ritz vectors, called "constraint modes" and "attachment modes," augment the component normal modes.‡ Although the basic idea can be generalized to permit the use of simple shape vectors rather than normal modes,[12] a systematic procedure for generating the vectors is required for practical applications. This paper, based on Refs. 13 and 14, develops such a procedure, referred to as the "block-Krylov method." The Ritz vectors employed are generated in a manner similar to that used (for complete structures, but not for components) in Refs. 15 and 16. In Ref. 17, Lanczos vectors are employed in dynamic substructure analysis in a manner similar to the block-Krylov procedure of this paper.

The four main objectives of this paper are the following:

1) Derive component recurrence relations that, when applied recurrently, generate Ritz vectors suitable for representing a component.

Both free-interface and fixed-interface recurrence relations are derived. Attachment modes, introduced by other authors (e.g., Refs. 2, 5–8) to improve the accuracy of free-interface component modal synthesis, appear naturally in the present free-interface recurrence relations. Likewise, constraint modes, defined in Refs. 1–4 for fixed-interface modal synthesis, appear in the present fixed-interface recurrence relations. The presence of each arises in a natural way in the corresponding derivation.

2) Discuss reduced-order component models based on Ritz vectors computed via the recurrence relations.

Repeated application of the recurrence relations generates a set of component Ritz vectors. The subspace spanned by the Ritz vectors is a block-Krylov subspace. It is a block subspace because Ritz vectors are generated in blocks of several vectors at once rather than one vector at a time. After the first block of Ritz vectors is computed, each additional block is relatively inexpensive to compute when compared to computing component normal modes. Each additional block requires only a back substitution for each vector in the block followed by mutual orthogonalization of the additional vectors. A reduced-order component model is constructed from the set of Ritz vectors so generated. All such reduced-order component models are then synthesized together to form a system model. Herein, this entire model/couple procedure is termed the block-Krylov method of component synthesis.

3) Compare the numerical accuracy of the block-Krylov method with that of the component modal synthesis method.

Numerical examples for fixed-interface component representations are presented and discussed. As far as the system's eigensolution is concerned, the accuracy of the fixed-interface block-Krylov method is seen to be comparable to that of the fixed-interface component modal synthesis method of Ref. 3.

4) Discuss the disturbability and observability properties of reduced-order component models obtained by the block-Krylov method.

Disturbability and observability criteria of general linear systems theory (e.g., Refs. 18 and 19) are shown to be equivalent to simple criteria based on block-Krylov subspaces. All degrees of freedom of block-Krylov reduced-order models are, therefore, disturbable (observable) from disturbance loads (observations) at component boundary degrees of freedom. The disturbability/observability properties of the block-Krylov method suggest its importance to high fidelity reduced-order modeling for actively controlled flexible structures; e.g., large space structures.

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‡In the CMS literature, the word "mode" refers to a Ritz-type displacement vector defined in some specific manner. Normal modes (eigenvectors) constitute a special case.

Constraint Modes, Attachment Modes, and Inertia Attachment Modes

The equations of motion of a typical (undamped) component may be written as

$$m^s \ddot{x}^s(t) + k^s x^s(t) = f^s(t) \tag{1}$$

where the superscript s identifies the number, or label, of the particular component, and where m^s, k^s, x^s, and f^s are the component's mass matrix, stiffness matrix, displacement vector, and force vector, respectively. (To simplify the notation, the superscript s will be dropped until it is needed later in the section on Component Synthesis.) The following partitioned forms of Eq. (1) will be useful in the following derivations:

$$\begin{bmatrix} m_{ii} & m_{ib} \\ m_{bi} & m_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{x}_i \\ \ddot{x}_b \end{Bmatrix} + \begin{bmatrix} k_{ii} & k_{ib} \\ k_{bi} & k_{bb} \end{bmatrix} \begin{Bmatrix} x_i \\ x_b \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_b \end{Bmatrix} \tag{2}$$

$$\begin{bmatrix} m_{ii} & m_{ie} & m_{ir} \\ m_{ei} & m_{ee} & m_{er} \\ m_{ri} & m_{re} & m_{rr} \end{bmatrix} \begin{Bmatrix} \ddot{x}_i \\ \ddot{x}_e \\ \ddot{x}_r \end{Bmatrix} + \begin{bmatrix} k_{ii} & k_{ie} & k_{ir} \\ k_{ei} & k_{ee} & k_{er} \\ k_{ri} & k_{re} & k_{rr} \end{bmatrix} \begin{Bmatrix} x_i \\ x_e \\ x_r \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_e \\ f_r \end{Bmatrix} \tag{3}$$

where the designations b, e, i, and r are defined below. The total number of degrees of freedom of the component is N, while the number of degrees of freedom of the subsets b, e, i, and r are N_b, N_e, N_i, and N_r, respectively.

To relate the proposed block-Krylov method of component synthesis to previously described methods of component synthesis, the notation of Ref. 11, which reviews component modal synthesis methods, will be followed. Figure 1a shows a cantilever beam divided into three components, while Fig. 1b identifies the interior (i) and boundary (b) degrees of freedom of a typical component.§ If the component has rigid-body freedom, the boundary coordinates may be subdivided into rigid-body (r) and redundant, or excess (e), coordinates.

In component modal synthesis the component displacement coordinates x are related to a reduced set of generalized coordinates p by the transformation

$$x = \Psi p \tag{4}$$

where Ψ is a matrix whose columns are Ritz vectors, or so-called component modes. The types of modes that make up Ψ are: constraint modes (including rigid-body modes), attachment modes, and normal modes. Rigid-body modes, constraint modes, and attachment modes of a beam component are illustrated in Fig. 2 by Ψ_r, Ψ_c, and Ψ_a, respectively. Inertia-relief attachment modes, illustrated by Ψ_m in Fig. 2, are also employed in some versions of component modal synthesis. In the block-Krylov method of component synthesis described in this paper, component normal modes are replaced by a set of Krylov vectors.

The rigid-body modes Ψ_r of a component, e.g., see Fig. 2, can be defined by

$$\begin{bmatrix} k_{ii} & k_{ie} & k_{ir} \\ k_{ei} & k_{ee} & k_{er} \\ k_{ri} & k_{re} & k_{rr} \end{bmatrix} \begin{bmatrix} \Psi_{ir} \\ \Psi_{er} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

where I is an $r \times r$ unit matrix.

§The "boundary" designation may be considered to apply to all component freedoms where forces are applied or where adjacent components are attached.

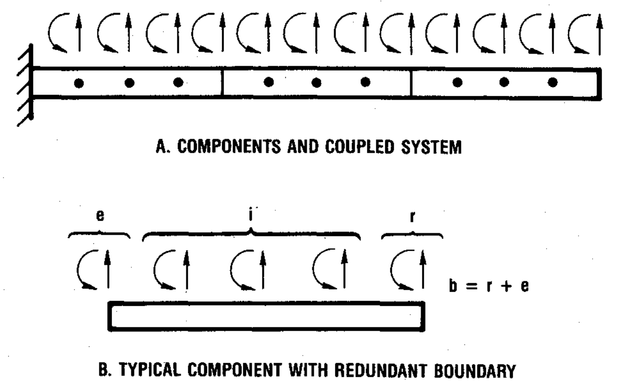

Fig. 1 The component modal synthesis method distinguishes between internal (i) and boundary (b) coordinates. Boundary coordinates are sometimes further divided into rigid-body (r) and excess (e), or redundant, coordinates.

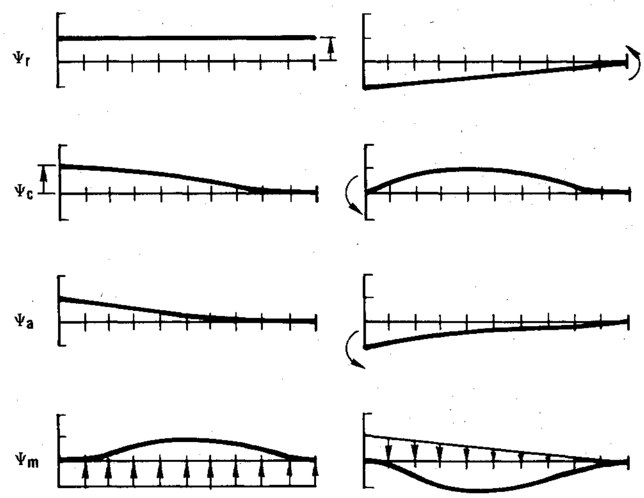

Fig. 2 Examples of component modes of a beam element.

Redundant constraint modes Ψ_h are defined by

$$\begin{bmatrix} k_{ii} & k_{ie} & k_{ir} \\ k_{ei} & k_{ee} & k_{er} \\ k_{ri} & k_{re} & k_{rr} \end{bmatrix} \begin{bmatrix} \Psi_{ih} \\ I \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ R_{ee} \\ R_{re} \end{bmatrix} \tag{6}$$

where I is an $e \times e$ unit matrix and R the reaction-force matrices.

The subscript h is used because these are the constraint modes used by Hurty in Ref. 1. A *constraint mode* is defined as the static deflection of a structure when a unit displacement is applied to one coordinate of a specified set of coordinates, while the remaining coordinates of that set are restrained and the remaining degrees of freedom of the structure are force free. In Ref. 3, it was shown that the rigid-body modes Ψ_r and redundant constraint modes Ψ_h span the same subspace as a set of constraint modes Ψ_c defined by

$$\begin{bmatrix} k_{ii} & k_{ib} \\ k_{bi} & k_{bb} \end{bmatrix} \begin{bmatrix} \Psi_{ic} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ R_{bb} \end{bmatrix} \tag{7}$$

That is, Ψ_c is given by

$$\Psi_c = \begin{bmatrix} \Psi_{ic} \\ I \end{bmatrix} = \begin{Bmatrix} -k_{ii}^{-1} k_{ib} \\ I \end{Bmatrix} \tag{8}$$

By definition, the set of constraint modes is "statically complete," i.e., the set is a basis for the linear space of all possible deformations that result from static loads applied at the boundary coordinates.

An *attachment mode* is defined as the static deflection of a structure when a unit force is applied at one coordinate of a specified set of coordinates, while the remaining unrestrained coordinates are force free. When a structure has rigid-body freedoms, either the structure can be restrained at an r-set of coordinates, or a set of inertia-relief attachment modes can be defined as in Refs. 6–8. The former approach leads to the following attachment modes Ψ_a defined for unit forces applied at the excess (redundant) coordinates.

$$\begin{bmatrix} k_{ii} & k_{ie} & k_{ir} \\ k_{ei} & k_{ee} & k_{er} \\ k_{ri} & k_{re} & k_{rr} \end{bmatrix} \begin{bmatrix} \Psi_{ia} \\ \Psi_{ea} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ I \\ R_{ra} \end{bmatrix} \tag{9}$$

Let

$$\begin{bmatrix} k_{ii} & k_{ie} \\ k_{ei} & k_{ee} \end{bmatrix}^{-1} \equiv \begin{bmatrix} g_{ii} & g_{ie} \\ g_{ei} & g_{ee} \end{bmatrix} \tag{10}$$

Then, from Eqs. (9) and (10)

$$\left[\Psi_a \equiv \begin{matrix} \Psi_{ia} \\ \Psi_{ea} \\ 0 \end{matrix} \right] = \begin{bmatrix} g_{ie} \\ g_{ee} \\ 0 \end{bmatrix} \tag{11}$$

It is shown in Refs. 7 and 11 that Ψ_a spans the same subspace as Ψ_h. Specifically,

$$\Psi_a = \Psi_h \, g_{ee} \tag{12}$$

Then, letting

$$\Psi_b \equiv [\Psi_r \Psi_a] \tag{13}$$

it is seen that Ψ_b and Ψ_c span the same $(N_r + N_e)$ subspace. Therefore, the set Ψ_b is "statically complete."

Hintz[7] defined a set of inertia-relief modes by the equation

$$\begin{bmatrix} k_{ii} & k_{ie} & k_{ir} \\ k_{ei} & k_{ee} & k_{er} \\ k_{ri} & k_{re} & k_{rr} \end{bmatrix} \begin{bmatrix} \Psi_{im} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m_{ii} & m_{ie} & m_{ir} \\ m_{ei} & m_{ee} & m_{er} \\ m_{ri} & m_{re} & m_{rr} \end{bmatrix} \begin{bmatrix} \Psi_{ir} \\ \Psi_{er} \\ I \end{bmatrix} + \begin{bmatrix} 0 \\ R_{em} \\ R_{rm} \end{bmatrix} \tag{14}$$

That is, the inertia-relief mode set Ψ_m is the set of static displacements of the component restrained at all boundary coordinates and loaded by the inertia forces resulting from unit acceleration in each of the rigid-body modes of Ψ_r. The inertia-relief modes Ψ_m are thus given by

$$\Psi_m \equiv \begin{bmatrix} \Psi_{im} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_{ii}^{-1} (m_{ii} \Psi_{ir} + m_{ie} \Psi_{er} + m_{ir}) \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

Examples of Ψ_m modes are shown in the bottom two figures of Fig. 2. References 7 and 11 show that the inertia-relief modes Ψ_m must be added to Ψ_b to form a "statically complete" mode set.

In Ref. 13, Craig proceeded to define a "sequence of inertia attachment modes" by the recurrence relation

$$\Psi_m^{(j+1)} \equiv \begin{bmatrix} \Psi_{im}^{(j+1)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_{ii}^{-1} \, m_{ii} \, \Psi_{im}^{(j)} \\ 0 \\ 0 \end{bmatrix}, \quad j = 1,2,\ldots \tag{16}$$

and a similar set of inertia attachment modes based on Ψ_h. As will be shown later, the "component fixed-interface block-Krylov subspace" defined by Hale[14] is the same space as that spanned by the vectors defined by Craig.[13]

Derivation of Component Recurrence Relations

To introduce component block-Krylov subspaces, it is convenient to show first how these subspaces arise naturally when recursive solution procedures are applied to structural dynamics problems. Over the years, various approximate representations of the component displacement vector have been proposed in the form of Ritz expansions, as in Eq. (4). The block-Krylov approximation proposed in this paper is suggested by recurrence relations, which may be derived from the component equations of motion. Although both eigenvalue and transient-response problems for the coupled system are of interest in structural dynamics, it is simpler to derive recurrence relations for the eigenproblem. In this case, the component must satisfy the equation

$$(k - \Omega^2 m)\bar{x} = \bar{f} \tag{17}$$

where Ω^2 is an eigenvalue of the assembled system. The corresponding system natural frequency is Ω, $\bar{x}$ is the portion of the system eigenvector within the component, and $\bar{f}$ constitutes boundary forces imposed on the component by adjacent components.

It is desired to manipulate Eq. (17) into forms in which $\bar{x}$ stands alone on the left-hand side. Two formulations are considered, depending on assumptions about the interface (boundary) degrees of freedom:

1) Interface degrees of freedom are constrained to be zero, i.e., a "fixed-interface" formulation.

2) All degrees of freedom, including those at the interface, are free, i.e., a "free-interface" formulation.

Note that although the assumptions above give the two formulations their names, the interface coordinates are neither "fixed" nor "free" in the final system synthesis, but, rather, are subjected to interelement compatibility conditions.

Fixed-Interface Recurrence Relations

Let Eq. (17) be written in partitioned form as follows:

$$\begin{bmatrix} k_{ii} & k_{ib} \\ k_{bi} & k_{bb} \end{bmatrix} \begin{Bmatrix} \bar{x}_i \\ \bar{x}_b \end{Bmatrix} = \Omega^2 \begin{bmatrix} m_{ii} & m_{ib} \\ m_{bi} & m_{bb} \end{bmatrix} \begin{Bmatrix} \bar{x}_i \\ \bar{x}_b \end{Bmatrix} + \begin{Bmatrix} 0 \\ \bar{f}_b \end{Bmatrix} \tag{18}$$

The top portion can be solved for $\bar{x}_i$ and the result combined with an identity for $\bar{x}_b$ to give

$$\bar{x} \equiv \begin{Bmatrix} \bar{x}_i \\ \bar{x}_b \end{Bmatrix} = \begin{bmatrix} -k_{ii}^{-1} k_{ib} \\ I \end{bmatrix} \bar{x}_b + \Omega^2 \begin{bmatrix} k_{ii}^{-1} m_{ii} & k_{ii}^{-1} m_{ib} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{x}_i \\ \bar{x}_b \end{Bmatrix} \tag{19}$$

It can safely be assumed that k_{ii} is nonsingular since the components will be connected by either statically determinate or redundant connections.

Equation (19) can be written in the abbreviated form

$$\bar{x} = \Psi_c \, \bar{x}_b + \Omega^2 G_c \, \bar{x} \tag{20}$$

where Ψ_c is the constraint mode matrix defined by Eq. (8) and G_c is defined by

$$G_c = \begin{bmatrix} k_{ii}^{-1} m_{ii} & k_{ii}^{-1} m_{ib} \\ 0 & 0 \end{bmatrix} \tag{21}$$

Based on Eq. (20), the following *fixed-interface recurrence formula* may be defined.

$$\bar{x}^{(j)} = \Psi_c \, \bar{x}_b^{(j)} + \Omega^2 \, G_c \, \bar{x}^{(j-1)}, \qquad j = 1,2,\ldots \tag{22}$$

Free-Interface Recurrence Relations

This case is complicated by the fact that, when all of the component degrees of freedom are free, the stiffness matrix k will be singular if the component is free to undergo rigid-body displacement. In such cases, a pseudoinverse of k is required rather than a regular inverse, and rigid-body modes must be included in the displacement of the component. This case will be assumed here for generality; the regular inverse suffices otherwise. A pseudoinverse can be defined in terms of the component's $(N-N_r)$ flexible free-free modes Φ_f and the diagonal matrix Λ_f of its nonzero eigenvalues, giving

$$k^{-1} = \Phi_f \Lambda_f^{-1} \Phi_f^T \tag{23}$$

Alternatively, the component may be (analytically) restrained at the rigid-body freedoms, and k^{-1} can be derived from Eq. (10) as

$$k^{-1} = \begin{bmatrix} g_{ii} & g_{ie} & 0 \\ g_{ei} & g_{ee} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{24}$$

The pseudoinverse defined by Eq. (24) will be employed in the following description.

For the free-interface case, let Eq. (17) be written in the expanded form

$$k(\bar{x}_r + \bar{x}_e' + \bar{x}_e'') = F_r\bar{f}_r + F_e'\bar{f}_e + F_e''\bar{f}_e + \Omega^2 m\bar{x} \tag{25}$$

where, in terms of the i, e, and r partitions of Eq. (3),

$$F_r = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad F_e' = \begin{bmatrix} 0 \\ I \\ R_{ra} \end{bmatrix}, \quad F_e'' = \begin{bmatrix} 0 \\ 0 \\ -R_{ra} \end{bmatrix} \tag{26}$$

and where $\bar{x}_r = \Psi_r \bar{p}_r$, and from Eqs. (9), (24), (25), and (26), $\bar{x}_e' = \Psi_a \bar{f}_e$. Then,

$$\bar{x} = \Psi_r\bar{p}_r + \Psi_a\bar{f}_e + \Omega^2 (k^{-1}m)\bar{x} \tag{27}$$

where k^{-1} is given by Eq. (24). Based on Eq. (27), the following *free-interface recurrence formula*, analogous to Eq. (22), may be defined.

$$\bar{x}^{(j)} = \Psi_r \bar{p}_r^{(j)} + \Psi_a \bar{f}_e^{(j)} + \Omega^2 (k^{-1}m)\, \bar{x}^{(j-1)}, \quad j = 1,2,\ldots \tag{28}$$

Equation (28) shows that any approximate representation of the component displacement should, as a minimum, contain the rigid-body modes Ψ_r and the attachment modes Ψ_a. That is, the solution should have Ψ_b, defined by Eq. (13), as a starting basis.

The above derivation of Eq. (28) is based on the pseudoinverse defined by Eq. (24). However, a similar derivation could be performed utilizing the pseudoinverse defined by Eq. (23) or some other pseudoinverse.

The Block-Krylov Method of Component Synthesis

Based on Eqs. (22), (28), and the descriptions leading up to these equations, a *component fixed-interface block-Krylov subspace* and a *component free-interface block-Krylov subspace* can be defined. Component synthesis is then discussed briefly, followed by several remarks on the numerical implementation of the method.

Definition of a Block-Krylov Subspace

A Krylov subspace of order $j(1 \le j \le n)$ is a j-dimensional vector space spanned by the columns of the matrix

$$P_j = [\phi, A\phi, A^2\phi, \ldots, A^{(j-1)}\phi] \tag{29}$$

where ϕ is any colum vector and A is a square matrix. We have assumed that ϕ is n-dimensional and A is $n \times n$-dimensional. Depending on the choice of ϕ and A, the basis vectors in Eq. (29) are either linearly dependent for some j less than n or they span the entire n-dimensional space when $j = n$.

If ϕ is replaced by a matrix with i columns rather than a single column, the subspace is called a block-Krylov subspace. In this case, at most only n of the total number $(i \times j)$ of columns of P_j can be linearly independent.

Component Fixed-Interface Block-Krylov Subspace

Starting with $\bar{x}^{(0)} = 0$, apply Eq. (22) repeatedly to obtain

$$\bar{x}^{(j)} = \Psi_c\, \bar{x}_b^{(j)} + \sum_{i=1}^{j-1} \Omega^{2i} G_c^i \Psi_c\, \bar{x}_b^{(j-i)} \tag{30}$$

This equation shows that $\bar{x}$ is a combination of the vectors in Ψ_c, $G_c\Psi_c$, $G_c^2\,\Psi_c$, etc. Thus, following Refs. 13 and 14, a component *fixed interface block-Krylov subspace* will be defined by

$$\Psi_j^c \equiv [\Psi_c, \Psi_c^{(1)}, \Psi_c^{(2)}, \ldots, \Psi_c^{(j-1)}] \tag{31}$$

where Ψ_c is given by Eq. (18) and where

$$\Psi_c^{(r)} \equiv \begin{bmatrix} \Psi_{ic}^{(r)} \\ 0 \end{bmatrix} = G_c^r\, \Psi_c \tag{32}$$

where G_c is defined by Eq. (21). Then,

$$\Psi_c^{(1)} \equiv \begin{bmatrix} \Psi_{ic}^{(1)} \\ 0 \end{bmatrix} = G_c\, \Psi_c = \begin{bmatrix} k_{ii}^{-1}\,(m_{ii}\,\Psi_{ic} + m_{ib}) \\ 0 \end{bmatrix} \tag{33}$$

and, subsequently,

$$\Psi_c^{(r)} = \begin{bmatrix} k_{ii}^{-1}\, m_{ii}\, \Psi_{ic}^{(r-1)} \\ 0 \end{bmatrix}, \qquad r = 2, 3, \ldots \tag{34}$$

Note that Eqs. (15) and (33) define the same inertia-relief mode set, while Eqs. (16) and (34) define the same sequence of inertia-relief mode sets. A physical interpretation of Eq. (34) is readily apparent. The deformation modes that characterize $\Psi_c^{(r)}$ have fully restrained boundary freedoms, and the interior displacement is the static deflection due to inertia loading associated with the preceding interior deflection shapes of $\Psi_{ic}^{(r-1)}$.

Component Free-Interface Block-Krylov Subspace

A free-interface analog of Eq. (31) can also be defined. First, let Eq. (28) be written in the more compact form

$$\bar{x}^{(j)} = \Psi_b\, \bar{p}_b^{(j)} + \Omega^2\, G_a\, \bar{x}^{(j-1)}, \qquad j = 1, 2, \ldots \tag{35}$$

where Ψ_b is defined by Eq. (13) and where

$$G_a = k^{-1}m \tag{36}$$

in which k^{-1} is defined by Eq. (24) and m by Eq. (3).

Starting with $\bar{x}^{(0)} = 0$, apply Eq. (35) repeatedly to obtain

$$\bar{x}^{(j)} = \Psi_b\, \bar{p}_b^{(j)} + \sum_{i=1}^{j-1} \Omega^{2i}\, G_a^i\, \Psi_b\, \bar{p}_b^{(j-1)} \tag{37}$$

Equation (37) shows that $\bar{x}$ may be considered to be a linear combination of the vectors in Ψ_b, $G_a\Psi_b$, $G_a^2\Psi_b$, etc. Let

$$G_a^r \Psi_b \equiv \Psi_b^{(r)} \tag{38}$$

with $\Psi_b^{(0)} \equiv \Psi_b$. Then,

$$\Psi_b^{(r)} = G_a^r\, \Psi_b = G_a\, \Psi_b^{(r-1)} = k^{-1}\, m\, \Psi_b^{(r-1)} \tag{39}$$

A physical interpretation of $\Psi_b^{(r)}$ is readily apparent from Eq. (39). The deformation modes that constitute $\Psi_b^{(r)}$ have constrained rigid-body (r) freedoms. The deflection at the remaining i and e freedoms is the static deflection due to the inertia loading associated with the preceding deflection shapes of $\Psi_b^{(r-1)}$.

Finally, the *free-interface block-Krylov subspace* of order j may be written as

$$\Psi_j^b = [\Psi_b, \Psi_b^{(1)}, \Psi_b^{(2)}, \ldots, \Psi_b^{(j-1)}] \tag{40}$$

Component Synthesis

The Ritz-type coordinate transformation (reduction) of Eq. (4) leads to the reduced-order component model

$$\mu \ddot{p} + \gamma \dot{p} + \kappa p = \Psi^T f \tag{41}$$

where¶

$$\mu = \Psi^T m \, \Psi, \qquad \gamma = \Psi^T c \, \Psi, \qquad \kappa = \Psi^T k \, \Psi \tag{42}$$

As an example of the structure of the preceding reduced-order component matrices, let

$$\Psi \equiv \Psi_2^c = [\Psi_c, \Psi_c^{(1)}, \Psi_c^{(2)}] \tag{43}$$

then

$$\kappa = \begin{bmatrix} \kappa_{11} & 0 & 0 \\ 0 & \kappa_{22} & \kappa_{23} \\ 0 & \kappa_{32} & \kappa_{33} \end{bmatrix} \tag{44}$$

where κ_{ij} represents a submatrix. The zeroes in Eq. (44) result from the k-orthogonality of the initial constraint modes in Ψ_c to all of the fixed-interface Krylov vectors which follow in $G_c \Psi_c$, etc. That is, from Eqs. (7) and (21),

$$\Psi_c^T \, k \, G_c \equiv 0 \tag{45}$$

The form of κ in Eq. (44) permits the direct stiffness method[21] of component coupling to be used to enforce the interface displacement compatibility equations for systems modeled by fixed-interface Krylov vectors.

To illustrate the procedure for coupling components to form a reduced-order system model, consider two components, α and β, and let the uncoupled component matrices be

$$p = \begin{bmatrix} p^\alpha \\ p^\beta \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu^\alpha & 0 \\ 0 & \mu^\beta \end{bmatrix}, \quad \kappa = \begin{bmatrix} \kappa^\alpha & 0 \\ 0 & \kappa^\beta \end{bmatrix} \tag{46}$$

Compatibility of interface displacements, and other linear constraint equations, may be collected to form the constraint equation

$$C p = 0 \tag{47}$$

References 10 and 11 indicate how the constraint matrix C can be used to define a coupling matrix S that, in turn, defines a set of independent system coordinates q, i.e.,

$$p = S q \tag{48}$$

Thus, the coupled system equations of motion for free vibration of an undamped system are

$$M \ddot{q} + K q = 0 \tag{49}$$

where

$$M = S^T \mu S, \qquad K = S^T \kappa \, S \tag{50}$$

Additional terms for damping or external forcing follow similarly.

¶A damping matrix is included in Eq. (41) for completeness, although this is not to suggest that this is necessarily the best approach for component synthesis for damped structures (e.g., see Ref. 20).

Remarks on Numerical Implementation

Several remarks on computing with block-Krylov vectors are appropriate here:

1) Numerical ill-conditioning may be encountered in forming the column vectors spanning a Krylov subspace. Consequently, some type of orthogonalization procedure should be incorporated into a practical implementation of the method. Because limited precision of numerical computations, the columns of ϕ, $A\phi$, $A^2\phi$, ... can be linearly dependent, even when they are theoretically independent. Therefore, in computations it is appropriate to orthogonalize the vectors as they are computed. This problem is accentuated when working with blocks of vectors. Fortunately, the problem is well-known and has been addressed in the literature on the Lanczos method (e.g., Ref. 22). Further research is being conducted on this problem as it relates to component synthesis.

2) Computation of block-Krylov vectors should not be done by inverting matrices. The expressions for G_c, Eq. (21), and G_a, Eq. (36), indicate the need for matrix inversion. Of course, implementing these expressions requires only a single factorization of the appropriate stiffness matrix.

3) Although numerical examples suggest that only a few blocks should suffice to give high accuracy in many applications, the question of how many blocks to include to obtain a given accuracy requires further research. A participation matrix similar to the one defined by Nour-Omid and Clough[16] may provide a suitable criterion for terminating the process of creating new blocks of Krylov vectors.

4) Although the question of how many blocks to include for a given accuracy remains, the starting point is well-defined. It has been shown that constraint modes (attachment modes) serve as the starting blocks for the fixed-interface (free-interface) block-Krylov method. In fact, those component modal synthesis methods that compute approximate modes via subspace iteration starting from random vectors should, instead, consider these blocks as a starting point for their iterations. This starting point is also appropriate to the multilevel iterative eigensolution methodology of Refs. 23 and 24, from which Ref. 14 evolved.

Disturbability/Observability Properties of Block-Krylov Models

This section is devoted to discussing the attractive disturbability and observability properties of reduced-order component models based on the block-Krylov method. Indeed, the Ritz vectors of Eqs. (31) and (40) have a special linear-system-theoretic interpretation. Namely, they arise when considering disturbability and/or observability of internal states from boundary disturbances/measurements. Hence, they are very good at representing component behavior when the primary interest is in the response of boundary degrees of freedom due to loadings at these degrees of freedom. Such interest is usually the case in launch vehicle/payload dynamic-loads models. Moreover, the special properties of the block-Krylov method suggest its importance to high-fidelity reduced-order modeling of actively controlled flexible structures, e.g., large space structures.

General Disturbability/Observability Criteria

Controllability/observability criteria for a linear, time-invariant system are well-known. Moreover, the concept of disturbability is related to that of controllability. Consider the system state equations written in first-order form:

$$\dot{x} = \underline{A} \, x + \underline{B} \, f \tag{51}$$

$$y = \underline{C} \, x \tag{52}$$

where x is the n-dimensional state vector, f is a p-dimensional vector of inputs (controls or disturbances), and y is a q-dimensional vector of outputs (measurements). The under-bar denotes a state matrix. The following are equivalent [e.g., see Refs. 18 and 19 for (1) and (3)]:

1) The triple $(\underline{A}, \underline{B}, \underline{C})$ is controllable and observable.
2) The triple $(\underline{A}, \underline{B}, \underline{C})$ is disturbable and observable.
3) The matrices $\underline{D}_n$ and $\underline{O}_n$ defined below have rank n.

$$\underline{D}_n = [\underline{B}, \underline{A}\,\underline{B}, \underline{A}^2\underline{B}, \ldots, \underline{A}^{n-1}\,\underline{B}] \tag{53}$$

$$\underline{O}_n = [\underline{C}^T, \underline{A}^T\underline{C}^T, \underline{A}^{2T}\underline{C}^T, \ldots, \underline{A}^{(n-1)T}\underline{C}^T] \tag{54}$$

In structural dynamics, the concept of disturbability is needed more often than that of controllability. Herein, a system is said to be disturbable if each and every state of the system can be excited by an input disturbance. On the other hand, controllability refers to the existence of an input f that forces (controls) $x(t)$ to the origin when starting from any finite initial state.

Disturbability/Observability Criteria for Undamped Substructures

For undamped substructures, which are governed by Eq. (1), specific disturbability and observability criteria can be derived in terms of the mass and stiffness matrices. For simplicity, frequency-domain equations like Eq. (17) will be used in the derivation. Furthermore, a free-interface formulation with a nonsingular stiffness matrix k will be considered.

Let Eq. (17) be rewritten in the form

$$\bar{x} = (I - \Omega^2\, k^{-1}\, m)^{-1}\, k^{-1}\, F_b\, \bar{f}_b \tag{55}$$

where F_b is the $(N \times N_b)$ boundary force distribution matrix

$$F_b = \begin{bmatrix} 0 \\ I_{bb} \end{bmatrix} \tag{56}$$

and let

$$\bar{y} = C_b\, \bar{x} \tag{57}$$

where C_b is the $(N_q \times N)$ measurement appropriation matrix. A Taylor series solution for $\bar{x}$, based on Eq. (55), is

$$\bar{x} = [I + \Omega^2\, k^{-1}\, m + (\Omega^4/2)\, (k^{-1}\, m)^2 + \cdots] k^{-1} F_b \bar{f}_b \tag{58}$$

This can be substituted into Eq. (57) to yield a Taylor series expansion for the measurement vector, namely

$$\bar{y} = C_b\, [I + \Omega^2\, k^{-1}\, m + (\Omega^4/2)\, (k^{-1}\, m)^2 + \cdots] k^{-1} F_b \bar{f}_b \tag{59}$$

The following proposition is based on Eqs. (58) and (59).

Proposition

An undamped N-degree-of-freedom component is completely disturbable from boundary excitation and completely observable from the observations of Eq. (57) if the respective matrices

$$D_n = [k^{-1}F_b, (k^{-1}m)\, k^{-1}F_b, \ldots, (k^{-1}m)^{N-1}\, k^{-1}\, F_b] \tag{60}$$

$$O_n = [k^{-1}C_b^T, (k^{-1}m)\, k^{-1}C_b^T, \ldots, (k^{-1}m)^{N-1}\, k^{-1}C_b^T] \tag{61}$$

have rank N. If D_n does not have rank N, the disturbable subspace is the linear space spanned by the columns of D_n. Similarly, if O_n does not have rank N, the unobservable subspace is the null space of O_n^T.

The Appendix provides a proof of the disturbability part of the preceding proposition; a proof of the observability part is similar.

Equations (60) and (61) define block-Krylov subspaces. They define the same subspace when all of the boundary degrees of freedom are observed, i.e., when $C_b^T = F_b$. In this case, all of the degrees of freedom of block-Krylov reduced-order models are disturbable (observable) from disturbance loads (observations) at component boundary degrees of freedom.

Numerical Examples

Two simple examples are presented in this section. They both compare the new block-Krylov method with the fixed-interface component-modal-synthesis method.[1,3] The first example considers a simple free-free Bernoulli–Euler beam that is divided into two components. The second example, Fig. 3a, considers a free-free planar truss divided into three components, Fig. 3b.

Free-Free Bernoulli–Euler Beam

The lower free-free natural frequencies for a uniform Bernoulli–Euler beam are compared in Table 1. The beam is divided into two components. The first component is 0.6 of the beam's length, and the second is the remaining 0.4 of the length. In this example the bending stiffness, the total length, and the mass per unit length are all taken to be unity. The first component is modeled by six equal-length finite elements; the second is modeled by four equal-length elements. In the table, B-K denotes the block-Krylov results and CMS denotes the Hurty-Craig-Bampton results. For B-K, $j+1$ is the total number of blocks; for CMS, n indicates the number of fixed-interface modes included for each component. Since there are two degrees of freedom at the boundary of each component with the other, namely, a transverse displacement coordinate and a rotational coordinate, each of the B-K blocks has two columns and each CMS component has two constraint modes. Therefore, the assembled systems with natural frequencies given in columns 1 and 2 have 6 degrees of freedom $(4+4-2=6)$, whereas those systems with frequencies given in columns 3, 4, and 5 have 10 degrees of freedom $(6+6-2=10)$.

Table 1 Comparison of block-Krylov and CMS (Hurty-Craig-Bampton) methods: Free-free natural frequencies (Hz) of example uniform beam

B-K $j=1$	CMS $n=2$	B-K $j=2$	CMS $n=4$	B-K/CMS $j=1/n=2$	Full FEM
0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	0.000	0.000
22.378	22.389	22.373	22.374	22.373	22.373
62.978	61.756	61.674	61.678	61.674	61.674
145.139	132.669	120.974	120.988	120.935	120.911
403.202	326.761	202.521	200.390	210.686	199.893
—	—	323.662	200.305	319.779	298.666
—	—	528.553	434.930	517.137	417.287

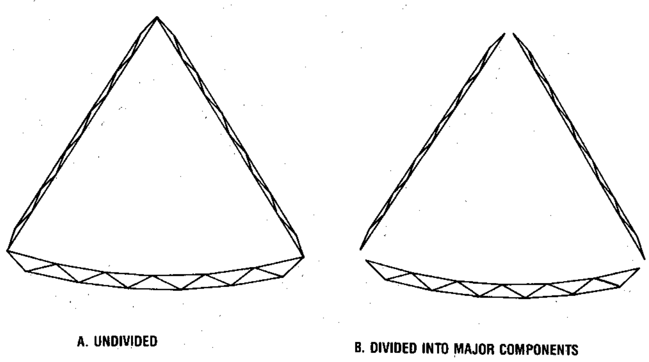

Fig. 3 The accuracy of the component block-Krylov method is compared to that of the CMS method using a planar truss antenna model made of three components.

Table 2 Comparison of block-Krylov and CMS (Hurty-Craig-Bampton) methods: Free-free natural frequencies (Hz) of example planar truss

B-K $j=1$	CMS $n=4$	B-K $j=2$	CMS $n=8$	Full FEM
0.000000E+0	0.000000E+0	0.000000E+0	0.000000E+0	0.000000E+0
0.000000E+0	0.000000E+0	0.000000E+0	0.000000E+0	0.000000E+0
0.000000E+0	0.000000E+0	0.000000E+0	0.000000E+0	0.000000E+0
1.953625E−2	1.953688E−2	1.953624E−2	1.953637E−2	1.953624E−2
2.205065E−2	2.205225E−2	2.205063E−2	2.205091E−2	2.205063E−2
4.923845E−2	4.926930E−2	4.923537E−2	4.924178E−2	4.923537E−2
7.273779E−2	7.231061E−2	7.229328E−2	7.229653E−2	7.229319E−2
7.626492E−2	7.570881E−2	7.567773E−2	7.568362E−2	7.567759E−2
1.550032E−1	1.528360E−1	1.528202E−1	1.528279E−1	1.528184E−1
1.617472E−1	1.586785E−1	1.586222E−1	1.586326E−1	1.586193E−1

As seen by comparing the columns of Table 1, the new method compares very well with component modal synthesis. This is easily seen by comparing those system models obtained by each method that have equal numbers of component degrees of freedom, i.e., by comparing columns 1 and 3 with columns 2 and 4, respectively. Moreover, by comparing these columns with the last column, which contains the natural frequencies computed for the full-order (22 DOF) finite-element model, the block-Krylov method is seen to be slightly more accurate for the lowest system modes, and the component-modal-synthesis method is seen to be slightly more accurate for the higher system modes. The fifth column shows natural frequencies for a cross between the block-Krylov and the modal-synthesis methods in which the block-Krylov method with $j=1$ is supplemented by the two lowest fixed-interface component normal modes. From the results, no advantage is seen in this cross formulation.

Planar Truss-Antenna Model

The truss structure shown in Fig. 3a is a planar model representing a large truss-antenna. Although the model is not associated with a specific antenna application, it is, nevertheless, useful for numerical comparisons. It is divided into three components as shown in Fig. 3b, namely, two feed masts and a reflector.

The natural frequencies of various assembled models are given in Table 2. Since the truss is planar, each finite-element nodal point has two degrees of freedom. Hence, each of the three components has four boundary degrees of freedom. The blocks of the block-Krylov method then have four columns, whereas each component of the modal-synthesis method has four constraint modes. As noted previously, B-K denotes the block-Krylov method with $j+1$ being the number of blocks; CMS denotes the modal-synthesis method with n being the number of fixed-interface normal modes for each component. Assembled systems with natural frequencies given in columns 1 and 2 have 18 degrees of freedom (DOF), whereas systems with frequencies given in columns 3 and 4 have 30 DOF. The full-order finite-element model has 72 DOF. As indicated earlier, the block-Krylov method is seen to have accuracy comparable to the modal-synthesis method. When comparing frequencies for assembled systems with equal numbers of degrees of freedom, the block-Krylov method once again yields slightly more accurate lower system frequencies than the modal-synthesis method. The modal-synthesis method, on the other hand, yields slightly more accurate higher system frequencies than the block-Krylov method does. Finally, note that the block-Krylov method accurately predicts the lowest seven nonzero system natural frequencies using only two blocks for each component, i.e., with $j=1$.

Conclusions

The block-Krylov method is a simple and accurate method for generating a set of component Ritz vectors for use in component synthesis. The block-Krylov Ritz vectors are generated from easily derived recurrence relations. The vectors span a subspace that is disturbable (observable) from disturbances (measurements) at the component's boundary coordinates. Because of the disturbability and observability properties, the method is particularly suited to applications requiring high-fidelity reduced-order models. One such application area is that of large space structures. The accuracy of the fixed-interface block-Krylov method has been shown via a numerical example to be comparable to that of fixed-interface component modal synthesis. Since the computational expense of the block-Krylov method is less than that for component modal synthesis, the block-Krylov method is preferable.

Appendix: Disturbability/Observability Criteria

This appendix sketches a proof of the disturbability proposition of Eq. (60); a proof of the observability proposition is similar.

Note that Eq. (58) uses the fact that $\bar{x}$ can be written as a linear combination of the block matrices in Eq. (60) and an infinite number of additional blocks. However, because $\bar{x}$ is only N-dimensional, there must eventually be a matrix

$$\Phi_L = (k^{-1}\, m)^L\, k^{-1}\, F_b \tag{A1}$$

with columns that are linearly dependent on the combined column vectors of the preceding matrices Φ_j [$j=0, 1, \ldots, (L-1)$]. There must be such a matrix since there cannot be more than N linearly independent vectors in the N-dimensional space. This implies that $L \le N$; in fact, it implies that $L \le \bar{N}$, where $\bar{N} = N/N_b$, or, if N/N_b is not an integer, then the next integer larger than N/N_b. Since the columns of Φ_L depend linearly on the columns of Φ_j [$j=0, 1, \ldots, (L-1)$], the following expression for Φ_L may be written,

$$\Phi_L = \Phi_0\, H_0 + \Phi_1\, H_1 + \cdots + \Phi_{(L-1)}\, H_{(L-1)} \tag{A2}$$

where the H are appropriate coefficient matrices. It follows that each matrix Φ_j ($L \le j \le \bar{N}$) depends linearly on the same column vectors. Therefore, $\bar{x}$ is a linear combination of the columns of D_L. That is, each excitation $\bar{f}_b$ can only produce a response $\bar{x}$ dependent on the columns of D_L. Moreover, all directions (columns) can be excited; i.e., disturbed. Complete disturbability follows for those components for which the rank of D_L equals N.

The disturbability/observability criteria of Eqs. (60) and (61) may also be related to the classical form of disturbability and observability in Eqs. (53) and (54), respectively. By defining the component state vector $\boldsymbol{x}$ to be

$$\boldsymbol{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{A3}$$

Equation (1) for an undamped structural component can be written in the form of Eq. (51), where

$$\underline{A} = \begin{bmatrix} 0 & I \\ -m^{-1}k & 0 \end{bmatrix}, \qquad \underline{B} = \begin{bmatrix} 0 \\ m^{-1}F_b \end{bmatrix} \tag{A4}$$

where $f^s(t) = F_b f_b(t)$ with F_b defined by Eq. (56). Expanding Eq. (53) using Eqs. (A4) yields

$$\underline{D}_L = \begin{bmatrix} 0 & m^{-1}F_b & 0 & (-m^{-1}k)m^{-1}F_b & \cdots & 0 & (-m^{-1}k)^{(L-1)}m^{-1}F_b \\ m^{-1}F_b & 0 & (-m^{-1}k)m^{-1}F_b & 0 & \cdots & (-m^{-1}k)^{(L-1)}m^{-1}F_b & 0 \end{bmatrix} \tag{A5}$$

A similar expression can be found for $\underline{O}_L$. Equation (A5) shows that x and $\dot{x}$ are both spanned by the column vectors of

$$d_L = [m^{-1}F_b, (m^{-1}k)\, m^{-1}F_b, \ldots, (m^{-1}k)^{(L-1)}\, m^{-1}F_b] \tag{A6}$$

If k is nonsingular, Eq. (A6) can be multiplied by $(k^{-1}m)^L$ to yield

$$D_L = [(k^{-1}m)^{(L-1)}\, k^{-1}F_b, \ldots, (k^{-1}m)\, k^{-1}F_b, k^{-1}F_b] \tag{A7}$$

whose blocks are defined in the same manner as the blocks in the disturbability criterion, Eq. (60).

It remains to be shown that the columns of d_L and D_L span the same subspace. To show that they do, define

$$\Psi_i = (m^{-1}k)^i m^{-1}F_b, \qquad \Phi_i = (k^{-1}m)^i k^{-1}F_b \tag{A8}$$

As before, there is an $L \leq \bar{N}$ such that

$$\Phi_L = \sum_{i=0}^{L-1} \Phi_i H_i \tag{A9}$$

and such that each Φ_j $(L \leq j \leq \bar{N})$ depends linearly on the same column vectors Φ_i $[i = 0, 1, \ldots, (L-1)]$. It will be shown that each Φ_j $[0 \leq j \leq (L-1)]$ also depends linearly on the columns of Ψ_i $[i = 0, 1, \ldots (L-1)]$, and vice versa. For brevity, only the former is demonstrated here.

First, let Φ_L be premultiplical by $(m^{-1}k)^L$ to yield

$$(m^{-1}k)^L \Phi_L = \Phi_0 = \sum_{i=0}^{L-1} \Psi_i H_{(L-1-i)} \tag{A10}$$

Equation (A10) shows that Φ_0 is a linear combination of all of the columns of Ψ_i $[0 \leq i \leq (L-1)]$. Next, let Φ_L be premultiplied by $(m^{-1}k)^{L-1}$ to yield

$$\begin{aligned} \Phi_1 &= \Phi_0 H_{(L-1)} + \sum_{i=0}^{L-2} \Psi_i H_{(L-2-i)} \\ &= \sum_{i=0}^{L-1} \Psi_i H_{(L-1-i)} H_{(L-1)} + \sum_{i=0}^{L-2} \Psi_i H_{(L-2-i)} \end{aligned} \tag{A11}$$

Thus, Φ_1 is also a linear combination of the vectors in Ψ_i $[0 \leq i \leq (L-1)]$. The argument may be continued until it is shown that each Φ_i $[i = 0, 1, \ldots, (L-1)]$ is a linear combination of the columns of the Ψ_j $[j = 0, 1, \ldots, (L-1)]$. Therefore, the linear space spanned by the columns of D_L is contained in the space spanned by the columns of d_L. Similar arguments show that the span of d_L is contained in the span of D_L. Hence, d_L and D_L both span the same subspace. Hence, the disturbable subspace is spanned by the columns of d_L and also by the columns of D_L. Of course, similar conclusions may be found for observability. Finally, the disturbable and observable subspaces are invariant under $m^{-1}k$ (for D_L) and $k^{-1}m$ (for d_L).

The above results are perhaps not fully appreciated within the structural dynamics and control of flexible structures communities. The results have important implications regarding the approximation of disturbable/observable subspaces by subspaces of lower dimension. If, in the Rayleigh-Ritz method, a basis includes the column vectors of D_0, no approximation is introduced into solutions for given static loads. This is not true, however, if the basis includes only a reduced number of the columns of d_L. The subspaces spanned by the columns of D_i are accurate from zero frequency at $i = 0$ to progressively higher frequencies for $i \geq 1$, while the subspaces d_i are accurate from infinity at $i = 0$ to progressively lower frequencies for $i \geq 1$. Since the structural model itself is only accurate in the lower frequency region, reduced-order models based on the former are preferable.

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